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Book Reviews: *Two from Sinai*

Theory of Phase Transitions: Rigorous Results. By Ya. G. Sinai. Pergamon Press, Oxford, 1982.

Ergodic Theory. By I. P. Cornfeld, S. V. Fomin, and Ya. G. Sinai. Springer Verlag, New York, 1982.

It is a pleasure to have two such excellent books to review. They deal, respectively, with central issues of equilibrium and nonequilibrium statistical mechanics.

1. The phase diagram of a system gives the decomposition of its thermodynamic parameter space into regions in which the number of pure phases (extremal periodic Gibbs states) is constant. In a series of papers, now beautifully described in Chapter 2 of Sinai's book, Pirogov and Sinai developed a comprehensive description of the low-temperature phase diagram for a large class of lattice systems: systems used to model Ising-like magnetic transitions as well as coherent transitions in crystalline alloys, i.e., ones in which the crystal structure of the alloy remains unchanged.

The setup is as follows: The system is described by occupation (or spin) variables which can take on a finite number of values at each site of a d-dimensional regular lattice, $d \geqslant 2$. The particles can interact with arbitrary finite range periodic potentials, e.g., a spin-$1/2$ Ising system with one, two, and three spin interactions. The Hamiltonian, H_0, has n periodic ground states, n finite. There is a nonzero minimum energy per unit interface, or "contour," separating two ground states: the Peierls' condition.

Pirogov and Sinai study the structure of the phase diagram of the Hamiltonian H_μ,

$$H_\mu = H_0 + \sum_{i=1}^{n} \mu_i H_i \tag{1}$$

in the n-dimensional parameter space $\mu_1, \ldots, \mu_n$. They prove that at sufficiently low temperatures the phase diagram perfectly mimics the topological structure of the ground states of H_μ: There are n-lines emanating

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from the origin on which H_μ has $n-1$ periodic ground states, two-dimensional surfaces bounded by pairs of these lines on which there are $n-2$ ground states, etc.

As an illustration consider the case of a spin—one system on a cubic (or other) lattice with nearest-neighbor interactions:

$$H_0 = J \sum_{\langle i,j \rangle} (S_i - S_j)^2, \qquad S_i = -1, 0, 1, \quad J > 0$$

$$H_\mu = H_0 - \mu_1 \sum S_i - \mu_2 \sum S_i^2$$

Then the structure of the phase diagram at zero and low temperatures can be obtained from the Pirogov–Sinai theory. It is sketched in Figs. 1 and 2. The symbols I, II, and III refer in Figure 1 to the ground states $S_i = -1$, $S_i = +1$, and $S_i = 0$, all i, and in Figure 2 to the corresponding pure phases. The bold lines represent phase boundaries. The uniqueness of phase III at $\mu_1 = \mu_2 = 0$ in Figure 2 is due to the fact that its entropy is higher; it has twice as many low-energy excitations, corresponding to changing a single-spin S_i $|\Delta S_i| = 1$, per unit volume. The shape of the lines in Figure 2 can be obtained as an asymptotic expansion in $\exp[-J/k_B T]$. (They are not drawn accurately in the figure.)

The physics behind this picture is simple: owing to the Peierls' condition, the low-temperature pure phases are nothing more than ground states in which there is a "sprinkling of droplets" of the other ground states. The contours surrounding these droplets represent excitations. At low temperatures the high energy cost of contours with large areas dominates the entropy and keeps them small and dilute. The Pirogov–Sinai theory may be

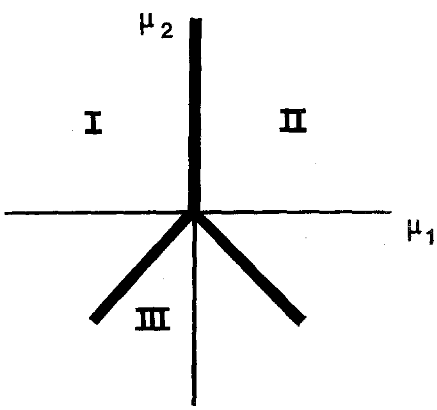

Fig. 1. $T = 0$.

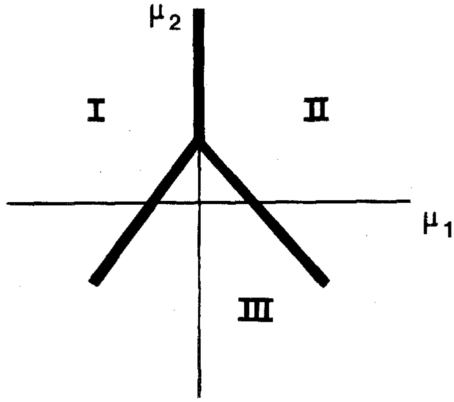

Fig. 2. $0 < T \ll J/k_B$.

thought of as an extension of the Peierls' argument for ferromagnetic Ising spins with nearest-neighbor interactions to systems in which the Hamiltonian and ground states do not possess any symmetries. The rigorous mathematical proof of this fact is highly nontrivial and involves some powerful mathematics.

This theory is, as far as we know, the only rigorous theory which deals with phase transitions in general lattice systems, but unfortunately it is not well known to physicists and metallurgists despite the simplicity of its concepts and the significance of its conclusions. The appearance of this book, based on lectures Sinai gave in Hungary in the late 1970s and updated with additional comments, should at least partially remedy this situation. We say partially because the book is not easy casual reading. The arguments are mathematically deep and require concentration—an effort the reader will not regret. We should also add that the book is extremely well organized and written. The reader interested only in the results and general arguments will find what he wants without going through all the details.

The other three chapters, each one being fairly self contained, are devoted to the following topics:

The first chapter deals with the description of an equilibrium system by a Gibbs probability distribution on its phase space [i.e., one of the form $\exp(-\beta H)$, where H is the Hamiltonian of the system]. In order to have a sharp notion of phase transition one is led to study systems with infinite spatial extension (the so-called thermodynamic limit). This introduces various problems since in the infinite system the Hamiltonian, being a sum of local interactions between the spins, is itself infinite. The solution of this problem, due to Dobrushin, Lanford, and Ruelle, is first to consider the set of all Gibbs distributions in all finite volumes V with all possible boundary conditions (b.c.), i.e., fixed configurations outside V. Then a probability distribution on the phase space of the infinite system is a Gibbs state (i.e., describes a situation of thermal equilibrium) if, for any finite V, its conditional probability distribution, given a configuration outside V (a b.c.), is the Gibbs distribution inside V with these b.c. Sinai introduces this fundamental concept and illustrates it with many examples. He discusses in detail the problem of the existence of these distributions with a prescribed system of conditional probabilities and lists some of their general properties.

The third chapter considers the situation where the set of spin values is no longer finite but is a space on which a continuous group acts, e.g., the classical Heisenberg model. Now, owing to the continuous symmetry, new phenomena occur. In particular, any phase transition is accompanied by long-range correlations between the spins. The absence of symmetry breaking in two dimensions and the existence of spontaneous magnetization in three dimensions are discussed. The Kosterlitz–Thouless transition is not.

Finally, Chapter 4 presents a rigorous renormalization group analysis of Dyson's hierarchical model. Again, the mathematical analysis is far from trivial but is carefully explained.

This book is highly recommended for researchers and graduate students with an interest in rigorous results. It can also be used for an advanced graduate course in (mathematical) statistical mechanics. It is the best introduction to the work being done at the present time, both East and West, on various extensions of the Pirogov–Sinai theory, e.g., to cases where there is an infinite degeneracy of the ground state and to continuum fluids.

2. "Ergodic theory is a powerful amalgam of methods used for the analyses of statistical properties of dynamical systems . . . *the problems of ergodic theory now interest not only the mathematician, but also the research worker in physics, biology, chemistry, etc.*" Thus begins the preface to this long and (by many) long-awaited book by Cornfeld, Fomin, and Sinai. We put the last part of the quote in italics since most nonmathematicians trying to understand the complicated behavior of nonlinear dynamical systems (which is more or less everything in the universe) hardly think that they are dealing with problems in ergodic theory. In fact many of them still think of ergodic theory as something invoked (like a credo or litany) in the first session of a course on statistical mechanics to justify the use of the microcanonical ensemble for isolated classical systems with given total energy. Like Moliere's gentleman, however, they have been talking prose all

the time. Their problems are very much the subject of ergodic theory and they can benefit much from its study.

In the description of ergodic theory given above the word "statistical" is crucial. There is always some stationary probability measure which determines what is significant and what is not. When, as usually happens, there are many stationary measures, it is the scientist's job to choose the right one for the physical situation modeled by the dynamical system. While the search for stationary measures is part of ergodic theory, the real mathematical fun begins after one has been chosen. Mathematical ergodic theory has undergone considerable growth and renewal in the last 25 years. Links have been developed with several branches of mathematics: probability theory, of course, but also group theory and number theory. There exist many books devoted to specific aspects of ergodic theory or to the general theory. However, none of them cover, to the extent done in this book, both the numerous applications and the variety of existing techniques.

The method followed by the authors is to start with examples and applications. The first chapter introduces the basic definitions (ergodicity, mixing, unitary operators associated to measure preserving transformations) and the main classical theorems (Birkhoff–Khinchin ergodic theorem, Krylov–Bogoliubov theorem on the existence of an invariant measure). Then come various special cases: Hamiltonian mechanics, translations of the torus, homeomorphisms of the circle, endomorphisms of compact groups, billiards, continued fractions, Gaussian stochastic processes, ideal gases, etc. Ergodic properties of these systems are discussed and the richness of the theory is displayed by these examples.

In the second part of the book one finds some general results of ergodic theory. The Kolmogorov–Sinai entropy is introduced and a proof is given of Ornstein's result on the isomorphism of Bernoulli shift with equal entropy.

The third part is devoted to the spectral properties of the unitary operators associated to measure-preserving transformations. It contains the von Neumann theory on ergodic automorphisms with pure point spectrum: the set of eigenvalues determines these automorphisms up to isomorphisms. The spectral theory of K-automorphisms and of Gaussian stochastic processes is also to be found there.

Finally, in the fourth part, the authors discuss approximation techniques of general dynamical systems by periodic ones. This is a topic of much current interest and is highly recommended to those doing numerical studies of dynamical systems.

While the book is generally well written, it is not exactly easy reading. It is therefore helpful that different sections are relatively autonomous, which facilitates browsing. It is likely to become a standard reference for everyone in the field.

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